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Conformally flat Lorentzian hypersurfaces in the conformal compactification of Lorentz space

M.P. Dussan[a], M. Magid[b,*]

[a] *Universidade de São Paulo, Departamento de Matemática -IME. CEP: 05508-090, São Paulo, Brazil*
[b] *Department of Mathematics, Wellesley College, Wellesley, MA 02458, United States*

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Abstract

We study conformally flat Lorentzian hypersurfaces in the conformal compactification of Lorentz space $\mathbb{R}_1^{n+1}$, which is the projectivized light cone $\widehat{\mathbb{R}}_1^{n+1} \subset \mathbb{R}P^{n+2}$ induced from $\mathbb{R}_2^{n+3}$. We establish a Lorentzian version of the local classification theorem of Cartan, in terms of branched channel hypersurfaces for $n \geq 4$, and for $n = 3$, in terms of the conformal fundamental forms. For hypersurfaces whose shape operator has complex eigenvalues, we give a necessary condition for being conformally flat in terms of local integrability of distributions.

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1. Introduction

Recent work of U. Hertrich-Jeromin established a new characterization of conformally flat hypersurfaces in the conformal sphere S^4 through the special triple orthogonal system known as Guichard's net (see [4,5]). That characterization gave a "demystification" of Cartan's Theorem, which established that the condition of conformal flatness was equivalent to the umbilicity of certain distributions [3]. Moreover, for dimensions greater than 3, U. Hertrich-Jeronim's work also involved a proof of the classical local characterization of conformally flat hypersurfaces in S^{n+1} given by Cartan, namely: A hypersurface $n \geq 4$ in the sphere is conformally flat if and only if it is a (branched) channel hypersurface. U. Hertrich-Jeromin's results involved a new method, namely, using the projective model for Möbius geometry for submanifolds in the Lorentz space $\mathbb{R}_1^{n+3}$, where the conformal sphere S^{n+1} was identified with the projectivized light cone in $\mathbb{R}P^{n+2}$.

* Corresponding author. Tel.: +1 781 283 3124.
E-mail address: mmagid@wellesley.edu (M. Magid).

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Continuing in this line, we start a study of conformally flat hypersurfaces in the Lorentzian setting. This means considering conformally flat Lorentzian hypersurfaces in the projectivized light cone $\hat{\mathbb{R}}_1^{n+1}$ in $\mathbb{R}P^{n+2}$ induced from $\mathbb{R}_2^{n+3}$. We observe that, in contrast to the positive definite case, where the projectivized light cone is identified with S^{n+1}, our projectivized light cone is identified with the compact space $\frac{S^n \times S^1}{S^o}$, which is called the conformal compactification of the Lorentz space $\mathbb{R}_1^{n+1}$ (see Porteous [11]). So, the main goal in this note is to prove a Lorentzian version of Cartan's Theorem for dimensions $n \geq 3$, in terms of the branched channel Lorentzian hypersurfaces, for the $n \geq 4$-dimensional case, and for dimension $n = 3$, in terms of the integrability of the conformal fundamental forms defined for the Lorentzian case. *In order to do this we restrict our attention to open sets on which the shape operators have constant algebraic types and the eigenvalues have constant multiplicities*. We assume these conditions throughout this note.

In order to establish our results, we note that a construction of the foundations of Möbius geometry of Lorentzian surfaces was made by the second author in ([8]) and the generalized definitions and basic facts for Lorentzian manifolds with dimension ≥ 2 can be extended naturally.

We recall that a pseudo-Riemannian manifold (M, g) is called *conformally flat* if, for any $x \in M$, there exists a neighborhood U of x and a function $u : U \to \mathbb{R}$ such that $(V, e^{2u} g)$ is flat.

Following Besse in [2], one finds that, as happens in the Riemannian case, any two-dimensional pseudo-Riemannian manifold is conformally flat. Analogously, for dimensions $n \geq 4$ a condition equivalent to conformal flatness is the vanishing of the Weyl tensor, and for $n = 3$ the criterion for conformal flatness is that the Schouten tensor is a Codazzi tensor.

We show in this paper the following result.

Lemma 1.1 (*Main Lemma*). *If the metric of a light cone representative $f : M_1^n \to \mathbb{R}_2^{n+3}$ of a Lorentzian hypersurface in the projectivized light cone $\hat{\mathbb{R}}_1^{n+1}$ is flat, then its normal bundle is flat (as an immersion into $\mathbb{R}_2^{n+3}$).*

In contrast to the Riemannian case, the shape operator in the Lorentzian case can have four possible forms, depending on its algebraic type: A_S is diagonalized over $\mathbb{R}$, or diagonalized over $\mathbb{C}$ (but not $\mathbb{R}$), or it is not diagonalizable with one eigenvalue of multiplicity 2 in the minimal polynomial corresponding to a null eigenvector, or with one eigenvalue of multiplicity 3 corresponding to a null eigenvector. (See [10], or [7] for details.) So, in the proof of Main Lemma 1.1, we use an f-adapted frame for the strip $(f, S) : M_1^n \to L_1^{n+2} \times S_2^{n+2}$, where S represents a spherical congruence in S_2^{n+2} enveloped by f, and study the four possible forms of the shape operator A_S. We show that the only case that cannot happen is that of multiplicity 3; for any other, the conditions imply the flatness of normal bundle in $\mathbb{R}_2^{n+3}$.

Moreover, using the ideas from [6], we prove a Lorentzian version of the theorem of Cartan's local characterization, namely:

Theorem 1.1. *$f : M_1^n \to \hat{\mathbb{R}}_1^{n+1}$, $n \geq 4$, is a conformally flat immersion iff f is a branched channel hypersurface.*

For the three-dimensional case, we first prove that all three-dimensional branched channel hypersurfaces are conformally flat and give explicit examples of three-dimensional conformally flat Lorentzian hypersurfaces which are not branched channel, including hypersurfaces whose shape operator has complex eigenvalues or is non-diagonal with one eigenvalue of multiplicity 2. We also study in more detail the so-called generic hypersurfaces, i.e., those whose shape operator is diagonalizable with three distinct real eigenvalues, or conjugate complex eigenvalues, or non-diagonalizable with one eigenvalue of multiplicity 2 or 3. In any case, we prove that the conformal flatness condition is equivalent to the *conformal fundamental forms* γ_i being closed. In particular this allows us to prove the following theorem:

Theorem 1.2. *If $f : M_1^3 \to \hat{\mathbb{R}}_1^4$ is conformally flat then the umbilic distributions $\gamma_i \pm \gamma_j = 0$ are locally integrable.*

Our results allow us to show the existence of special coordinate systems which, in the real and complex diagonal cases, represent the analogue of Guichard's nets.

This note is organized as follows. Section 2 contains the basic facts and Lorentzian definitions involving spherical congruences S, their envelopes f and adapted f-frames for the strip (f, S). In Section 3 we prove the Main Lemma using adapted f-frames. Section 4 is dedicated to studying the branched channel Lorentzian hypersurfaces in $\hat{\mathbb{R}}_1^{n+1}$

for $n \geq 4$. We prove Theorem 1.1 and in addition, we give explicit examples of three-dimensional Lorentzian generic hypersurfaces in $\hat{\mathbb{R}}^4_1$. Section 5 contains some results involving the local classification of the three-dimensional conformally flat Lorentzian hypersurfaces in $\hat{\mathbb{R}}^4_1$ and the proof of Theorem 1.2. Finally, in Appendix we establish the principal computations involving the Cartan tensor and the condition for the Schouten tensor to be a Codazzi tensor.

2. Spherical congruences

The principal goal in this section is to set the definitions and basic facts needed for proving the Main Lemma. We refer the reader to [8], where the second author studied the conformal geometry of Lorentzian surfaces, constructing the foundations of Möbius geometry for such surfaces. The definitions here for Lorentzian manifolds with dimensions greater than 2 are natural extensions.

Let $\mathbb{R}^{n+3}_2$ be $\mathbb{R}^{n+3}$ with the metric

$$\langle \vec{v}, \vec{w} \rangle = -v_1 w_1 + \sum_{i=2}^{n+1} v_i w_i + v_{n+2} w_{n+3} + v_{n+3} w_{n+2},$$

for $\vec{v} = (v_1, \ldots, v_{n+3})$, $\vec{w} = (w_1, \ldots, w_{n+3})$. We define a pseudo-orthonormal basis of $\mathbb{R}^{n+3}_2$, $\{e_1, \ldots, e_{n+3}\}$, as one such that $\langle e_i, e_j \rangle = \pm\delta_{ij}$ for $1 \leq i, j \leq n+1$, with -1 for $i = 1$ and $+1$ otherwise, and $e_{n+2}, e_{n+3} \in \{e_1, \ldots, e_{n+1}\}^{\perp}$ are null vectors with $\langle e_{n+2}, e_{n+3} \rangle = 1$. We also define an orthonormal basis $\{v_1, \ldots v_n\}$ of a Lorentzian n-dimensional space as one for which $\langle v_1, v_1 \rangle = -1$, $\langle v_i, v_j \rangle = \delta_{ij}$, and $\langle v_1, v_j \rangle = 0$ for $2 \leq i, j \leq n$.

Let $\mathbb{R}P^{n+2}$ denote the real projective space of lines passing through the origin in $\mathbb{R}^{n+3}$, π the projection from $\mathbb{R}^{n+3} - \{0\}$ to $\mathbb{R}P^{n+2}$, $L^{n+2}_1 = \{v \in \mathbb{R}^{n+3}_2 | \langle v, v \rangle = 0\}$ the light cone. Then the projection of $L^{n+2}_1 - \{0\}$ is homeomorphic to a compact space called *the conformal compactification of* $\mathbb{R}^{n+1}_1$ [11], and which we denote by $\hat{\mathbb{R}}^{n+1}_1$.

Following [8], one finds that there exists a bijection between points in $\mathbb{R}P^{n+2}$ and quadrics and planes in $\mathbb{R}^{n+1}_1$. More explicitly, spacelike points correspond to Lorentzian spheres and timelike planes, while the timelike points correspond to hyperbolic spaces and spacelike planes. Moreover, the Lorentzian spheres in $\mathbb{R}^{n+1}_1$ correspond to points in $\mathbb{R}P^{n+2}_+$, the projectivized spacelike points, while hyperbolic spaces correspond to points in $\mathbb{R}P^{n+2}_-$, the projectivized timelike points. Finally, points in $\mathbb{R}^{n+1}_1$ or S^{n+1}_1 or H^{n+1}_1 are identified with points in $\hat{\mathbb{R}}^{n+1}_1 = \mathbb{R}P^{n+2}_0$, the projectivized light cone in $\mathbb{R}P^{n+2}$ induced from $\mathbb{R}^{n+3}_2$.

For the Lorentzian case, the natural extensions of sphere congruence and its envelopes are given by the following definitions:

Definition 2.1 (*[8]*). A differential n-parameter family of spheres $S : M^n_1 \to \mathbb{R}P^{n+2}_+$ is called a spherical congruence. It corresponds to a family of Lorentzian spheres in $\mathbb{R}^{n+1}_1$.

Definition 2.2 (*[8]*). A differential map $f : M^n_1 \to \hat{\mathbb{R}}^{n+1}_1$ is called an envelope for the spherical congruence S if, for all $p \in M^n_1$, $f(p) \in S(p)$ and $T_{f(p)} f(M^n_1) \subset T_{f(p)} S(p)$.

A equivalent condition for being an envelope for a spherical congruence is given by the next lemma.

Lemma 2.1 (*[8]*). *A differential map $f : M^n_1 \to \hat{\mathbb{R}}^{n+1}_1$ envelopes a spherical congruence $S : M^n_1 \to \mathbb{R}P^{n+2}_+$ if and only if*

$$\langle f, S \rangle = 0 \quad \textit{and} \quad \langle \mathrm{d}f, S \rangle = 0.$$

By rescaling one can assume that S takes values in S^{n+2}_2, i.e., $S : M^n_1 \to S^{n+2}_2$ where the images are the unit spacelike vectors.

Finally, by analogy to the positive definite case, one can define a strip and an adapted frame, as follows.

Definition 2.3. A pair of smooth maps $(f, S) : M^n_1 \to L^{n+2}_1 \times S^{n+2}_2$, where f is an immersion and S is a spherical congruence enveloped by f, is called a strip.

Definition 2.4. Let $(f, S) : M_1^n \to L_1^{n+2} \times S_2^{n+2}$ be a strip and $\{e_i\}_{i=1}^{n+3}$ a pseudo-orthonormal basis of $\mathbb{R}_2^{n+3}$. A map $F : M_1^n \to O_2(n+3)$ such that $S = Fe_{n+1}$, $f = Fe_{n+2}$ and such that, for all $p \in M_1^n$, $\operatorname{span}\{Fe_1, \dots, Fe_n\}_p = \mathrm{d}f_p(T_pM_1^n)$ is called an f-adapted frame for the strip (f, S).

We note that any $\mathrm{e}^u f$, for some smooth function u on M_1^n, gives a conformally equivalent immersion.

Now we are ready to prove the Main Lemma.

3. Main Lemma's proof using f-adapted frames

In this section we prove the Main Lemma using an f-frame adapted for the strip (f, S) where f represents the immersion $f : M_1^n \to L_1^{n+2}$ and S, a spherical congruence enveloped by f. Since, in the indefinite setting, the shape operator A_S can have four different forms depending on the algebraic type, we must consider all these cases.

Proof of the Main Lemma. Let $f : M_1^n \to L_1^{n+2} \subset \mathbb{R}_2^{n+3}$ be a representative of the immersion into $\hat{\mathbb{R}}_1^{n+1}$ and $S : M_1^n \to S_2^{n+2} \subset \mathbb{R}_2^{n+3}$ be a spherical congruence enveloped by f. Now let F be a f-adapted pseudo-orthonormal framing for the strip (f, S), given by

$$F = (S_1, \dots, S_n, S, f, \widehat{f}) : M_1^n \to O_2(n+3)$$

with $\operatorname{span}\{S_1, \dots, S_n\}_p = \mathrm{d}f_p(T_pM_1^n)$ for all $p \in M_1^n$. In particular $S_i = Fe_i, i = 1, \dots, n, S = Fe_{n+1}, f = Fe_{n+2}$, and $\widehat{f} := Fe_{n+3}$. Moreover, $\{S_1 \dots, S_n\}$ forms an orthonormal set with $\langle S_1, S_1\rangle = -1$, with f and $\widehat{f}$ are null vectors such that $\langle f, \widehat{f}\rangle = 1$.

As usual $dFe_B = \sum_A \omega_{AB} Fe_A$. Then the connection form $\Phi = F^{-1}\mathrm{d}F : TM \to o_2(n+3)$ is given by

$$\Phi = \begin{pmatrix} \omega & \eta \\ -\eta^* & \nu \end{pmatrix} = \begin{pmatrix} \omega & \eta \\ -J'\eta^t I_{1,n-1} & \nu \end{pmatrix}$$

where $J' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,

$$\omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \dots & \omega_{1,n-1} & \omega_{1n} \\ \omega_{12} & 0 & \omega_{23} & \dots & \omega_{2,n-1} & \omega_{2n} \\ \omega_{13} & -\omega_{23} & 0 & \dots & \omega_{3,n-1} & \omega_{3n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \omega_{1n} & -\omega_{2n} & -\omega_{3n} & \dots & -\omega_{n-1,n} & 0 \end{pmatrix} : TM \to o_1(n)$$

$$\eta = \begin{pmatrix} -\psi_1 & -w_1 & -\zeta_1 \\ \psi_2 & w_2 & \zeta_2 \\ \vdots & \vdots & \vdots \\ \psi_n & w_n & \zeta_n \end{pmatrix} : TM \to \mathcal{M}(n \times 3) \quad \text{and} \quad \nu = \begin{pmatrix} 0 & 0 & v \\ -v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : TM \to o_1(3).$$

We observe that, since $\langle S, \mathrm{d}f\rangle = 0$ and $\langle \mathrm{d}f, \widehat{f}\rangle = 0$, one obtains $\omega_{n+1,n+2} = 0 = \omega_{n+3,n+1}$ and $\omega_{n+2,n+2} = 0 = \omega_{n+3,n+3}$. We write

$$\mathrm{d}f = \mathrm{d}Fe_{n+2} = \omega_{1,n+2}S_1 + \dots + \omega_{n,n+2}S_n = -w_1S_1 + \dots + w_nS_n.$$

Then the first and second fundamental forms are given by

$$\mathrm{I} = -w_1^2 + w_2^2 + \dots + w_n^2$$

$$\mathrm{II} = -\left(-w_1\psi_1 + \sum_2^n w_i\psi_i\right) S - \left(-w_1\zeta_1 + \sum_2^n w_i\zeta_i\right) f - \left(-w_1^2 + \sum_2^n w_i^2\right)\widehat{f}.$$

From now on we let $\tau_1 = -1$ and $\tau_i = 1$ for $2 \le i \le n$.

The integrability conditions for the existence of such a frame F, the Maurer–Cartan equations $\mathrm{d}\Phi + \frac{1}{2}[\Phi \wedge \Phi] = 0$, are the Gauss–Codazzi–Ricci equations for the immersion f, namely:

The Ricci equation: $\mathrm{d}\nu = \eta^* \wedge \eta$, i.e.,

$$\begin{cases} \mathrm{d}\upsilon = \sum_i^n \tau_i(\psi_i \wedge \zeta_i) \\ 0 = \sum_1^n \tau_i(\psi_i \wedge w_i) \\ 0 = \sum_1^n \tau_i(\zeta_i \wedge w_i), \end{cases} \tag{1}$$

the Codazzi equation: $\mathrm{d}\eta = -(\omega \wedge \eta + \eta \wedge \upsilon)$, which in components is

$$\begin{cases} \mathrm{d}\psi_i + \sum_j \tau_i\tau_j\omega_{ij} \wedge \psi_j = w_i \wedge \upsilon \\ \mathrm{d}w_i + \sum_j \tau_i\tau_j\omega_{ij} \wedge w_j = 0 \\ \mathrm{d}\zeta_i + \sum_j \tau_i\tau_j\omega_{ij} \wedge \zeta_j = \upsilon \wedge \psi_i, \end{cases} \tag{2}$$

and the Gauss equation $\rho = \eta \wedge \eta^*$ with the curvature form $\rho = \mathrm{d}\omega + \omega \wedge \omega$:

$$\rho_{ij} := \mathrm{d}\omega_{ij} + \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} = \tau_i(\psi_i \wedge \psi_j + w_i \wedge \zeta_j + \zeta_i \wedge w_j). \tag{3}$$

Then it follows from the second and third Ricci equations that the second fundamental forms $\mathrm{II}_S = -w_1\psi_1 + \sum_2^n w_i\psi_i$ and $\mathrm{II}_{\widehat{f}} = -w_1\zeta_1 + \sum_2^n w_i\zeta_i$, where S is being considered as a unit field normal to the immersion $f : M_1^n \to \mathbb{R}_2^{n+3}$, are symmetric forms with respect to the Lorentzian metric. Now we are interested in studying the shape operator in the S-direction A_S.

Lemma 3.1 (*Preliminary Lemma*). *If the shape operator has constant algebraic type and the eigenvalues have constant multiplicities in a neighborhood of a point x_o, then we can find a basis of vector fields in a neighborhood of x_o so that A_S has one of the four standard forms.*

Proof. We first look at the case where A_S has one pair of conjugate complex eigenvalues in a neighborhood of x_o. Thus we assume it has the following form at the point:

$$A_S = \begin{pmatrix} a_o & b_o & & & & \\ -b_o & a_o & & & & \\ & & a_1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & a_{n-2} \end{pmatrix}. \tag{4}$$

Pick an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ at one point, so that the first vector is unit timelike. Extend this to a basis of vector fields $\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n\}$. We consider these vector fields to be in the complexified tangent bundle $TM^{\mathbb{C}}$. Assume that the eigenvalues are $a \pm \mathrm{i}b, \lambda_1, \ldots \lambda_k$. We note that the distinct real eigenspaces $T_{\lambda_i}, T_{\lambda_j}$ are orthogonal, and if $v_1 + \mathrm{i}v_2$ is a complex eigenvector for $a + \mathrm{i}b$, then v_1 and v_2 are perpendicular to all T_{λ_j}. For $j = 1, \ldots, k$, by an abuse of notation, assume $u_j \in T_{\lambda_j}$ and set

$$t_j = (A - (a + \mathrm{i}b)I)(A - (a - \mathrm{i}b)I)(A - \lambda_1 I) \ldots (\widehat{A - \lambda_j I}) \ldots (A - \lambda_k I)\tilde{u}_j,$$

where the ˆ means the factor is omitted. At x_o we see that t_j is a non-zero multiple of u_j, and so remains non-zero in a neighborhood. Furthermore, note that $At_j = \lambda_j t_j$ and t_j are real vector fields.

Next set $t_1 = t_{11} + \mathrm{i}t_{12} = (A - (a - \mathrm{i}b)I)(A - \lambda_1 I) \ldots (A - \lambda_k I)\tilde{u}_1$. We note that $At_{11} = at_{11} - bt_{12}$ and $At_{12} = bt_{11} + at_{12}$. If we apply the Gram–Schmidt process to $t_n, t_{n-1}, \ldots, t_3, t_{12}, t_{11}$ we arrive at the desired basis of vector fields, with the final one of length -1.

We use essentially the same techniques for the multiplicity 2 case. We assume that at every point there is a basis of the form $\{L_1, L_2, u_3, \ldots, u_n\}$ with respect to which

$$A_S = \begin{pmatrix} a_o & \nu & & & & \\ 0 & a_o & & & & \\ & & a_1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & a_{n-2} \end{pmatrix},$$

L_1 and L_2 are null, $\langle L_1, L_2\rangle = 1$, $\langle L_i, e_j\rangle = 0$ and the u_i's form an orthonormal set. We choose such a basis at one point and extend it to a basis $\{\tilde{L}_1, \tilde{L}_2, \tilde{u}_3, \ldots, \tilde{u}_n\}$ and assume that the eigenvalues are $\lambda, \lambda_1, \ldots, \lambda_k$, where λ is the eigenvalue associated with the null eigenvector. For $j = 3, \ldots, n$ set

$$t_j = (A - \lambda I)^2 (A - \lambda_1 I) \ldots \widehat{(A - \lambda_j I)} \ldots (A - \lambda_k I)\tilde{u}_j.$$

Applying the Gram–Schmidt process to $\{t_n, \ldots, t_3, \frac{\tilde{u}_1+\tilde{u}_2}{\sqrt{2}}, \frac{\tilde{u}_1-\tilde{u}_2}{\sqrt{2}}\}$ to arrive at an orthonormal set $\{v_n, \ldots, v_3, w_2, w_1\}$. We see that $\{w_1, w_2\}$ is an orthonormal basis of $\{T_{\lambda_1} \oplus \cdots \oplus T_{\lambda_k}\}^{\perp}$ and that $w_2 - w_1$ is a null vector satisfying $(A - \lambda I)(w_2 - w_1) \neq 0$ at x_o. Since $(A - \lambda I)^2 (w_2 - w_1) = 0$ we see that $(A - \lambda I)(w_2 - w_1)$ is in T_λ and is a null vector field. To fill out the basis of vector fields we choose a null vector in the span of $\{w_1, w_2\}$ which is null and whose inner product with $(A - \lambda I)(w_2 - w_1)$ is 1.

The diagonal case and the final non-diagonal case can be handled in a similar fashion. □

We note that the proof for the diagonalized case is the same as the positive definite one and start our analysis assuming first that A_S has one pair of conjugate complex eigenvalues, i.e., A_S has the form (4) with respect to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of the tangent space at one point. Then we choose $w_1, \ldots, w_n$ so that $w_i(v_j) = \tau_i \delta_{ij}$. We can do this as follows: Assume that the orthonormal basis $\{u_1, \ldots, u_n\}$ is dual to the original choice $w_1, \ldots, w_n$. Assume further that an orthogonal transformation B takes u_j to v_j, i.e., $Bu_j = v_j$. Let $S_j^* = BS_j$. Then $\mathrm{d}f(v_j) = \mathrm{d}f(Bu_j) = B(\mathrm{d}f(u_j)) = B(S_j) = S_j^*$, so the new basis is dual to the new $w_1^*, \ldots, w_n^*$.

On the other hand, since $\mathrm{d}S(X) = -f_*(A_S(X)) + \nabla_X^{\perp} S$, it follows that for all i,

$$\begin{aligned} &-\psi_1(v_i)Fe_1 + \psi_2(v_i)Fe_2 + \cdots + \psi_n(v_i)Fe_n - v(v_i)Fe_{n+2} \\ &\quad = w_1(A_S(v_i))Fe_1 - w_2(A_S(v_i))Fe_2 - \cdots - w_n(A_S(v_i))Fe_n + \nabla_{v_i}^{\perp} S. \end{aligned} \tag{5}$$

Thus we get

$$\begin{cases} \psi_1 = -a_o w_1 + b_o w_2, \\ \psi_2 = -b_o w_1 - a_o w_2, \\ \psi_i = -a_{i-2} w_i, \quad i = 3, \ldots, n. \end{cases} \tag{6}$$

Now we assume the induced metric $\sum_i \tau_i w_i^2$ of the light cone representative f to be flat. Then the curvature forms ρ_{ij} in the Gauss equation vanish. Hence if we assume $\zeta_i = \sum b_{ik} w_k$, we get, from the Gauss equation $0 = \tau_i(\psi_i \wedge \psi_j + w_i \wedge \zeta_j + \zeta_i \wedge w_j)$,

$$\begin{aligned} &a_o^2 + b_o^2 + b_{11} + b_{22} = 0 \\ &a_o a_{j-2} + b_{ii} + b_{jj} = 0, \quad i = 1, 2,\ j \geq 3 \\ &-b_o a_{j-2} + b_{12} = 0, \quad j \geq 3 \\ &b_o a_{j-2} + b_{21} = 0, \quad j \geq 3 \\ &a_{i-2} a_{j-2} + b_{ii} + b_{jj} = 0, \quad 3 \leq i \neq j \leq n \\ &b_{ij} = 0 \quad i \neq j, i, j \neq 1, 2. \end{aligned}$$

So we have $b_{12} = b_o a_i$ for $1 \leq i \leq n-2$, and all a_i are equal for $1 \leq i \leq n-2$, since $b_o \neq 0$. Moreover, $b_{11} = b_{22}$ and all b_{ii} are equal for $3 \leq i \leq n$. Call these common values a and b respectively. Then

$$\begin{cases} \zeta_1 = b_{11} w_1 + b_{12} w_2 \\ \zeta_2 = -b_{12} w_1 + b_{11} w_2 \\ \zeta_i = b w_i, \quad i = 3, \dots, n. \end{cases} \tag{7}$$

Hence the second fundamental form with respect to $\widehat{f}$ has the same form as A_S, and they commute. Now substituting the values of ζ_i and ψ_i obtained above in the first Ricci equation, one gets that $\mathrm{d}v = 0$, i.e., the normal bundle of f is flat.

Now, we consider the case when the matrix A_S has one eigenvalue of multiplicity 2, i.e.,

$$A_S = \begin{pmatrix} \nu \pm \alpha & -\alpha & & & \\ \alpha & \nu \mp \alpha & & & \\ & & a_1 & & \\ & & & \cdot & \\ & & & & a_{n-2} \end{pmatrix}$$

with respect to an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of the tangent space. Then we choose $w_1, \dots, w_n$ such that $w_i(v_j) = \tau_i \delta_{ij}$. Next from A_S we read $A_S(v_i)$, $1 \leq i \leq n$, and use formula (5) to obtain

$$\begin{cases} \psi_1 = -(\nu \pm \alpha) w_1 - \alpha w_2, \\ \psi_2 = \alpha w_1 - (\nu \mp \alpha) w_2, \\ \psi_i = -a_{i-2} w_i, \quad i = 3, \dots, n. \end{cases} \tag{8}$$

Now assuming that the induced metric of the light cone representative f is flat, we get from the Gauss equation that

$$\begin{aligned} &\nu^2 + b_{22} + b_{11} = 0 \\ &b_{1j} = 0 = b_{2j}, \qquad b_{j2} = 0 \quad \text{for } j > 2 \\ &a_{j-2}(\nu \pm \alpha) + b_{11} + b_{jj} = 0 \quad \text{for } j \geq 3 \\ &\alpha a_{j-2} + b_{12} = 0 \quad \text{for } j \geq 3 \\ &b_{jk} = 0, \quad \text{for } j, k > 2, \\ &-\alpha a_{j-2} + b_{21} = 0 \quad \text{for } j \geq 3 \\ &b_{j1} = 0, \quad \text{for } j > 2 \\ &a_{j-2}(\nu \mp \alpha) + b_{22} + b_{jj} = 0 \quad \text{for } j \geq 3 \\ &a_{i-2} a_{j-2} + b_{ii} + b_{jj} = 0, \quad 3 \leq i \neq j \leq n. \end{aligned}$$

So we conclude that $b_{12} = -\alpha a_{j-2}$, i.e., all a_i are equal for $i = 1, \dots, n-2$ because we assume $\alpha \neq 0$. Call the common value a. In addition, $b_{21} = -b_{12}$ and $\pm 2a\alpha + b_{11} - b_{22} = 0$. We also get that all b_{jj} are equal for $j = 3, \dots, n$. So,

$$\begin{cases} \zeta_1 = b_{11} w_1 + b_{12} w_2 \\ \zeta_2 = -b_{12} w_1 + b_{22} w_2 \\ \zeta_i = b w_i, \quad i = 3, \dots, n. \end{cases} \tag{9}$$

But with those values of ζ_i and ψ_i the first Ricci equation gives us

$$\mathrm{d}v = -\psi_1 \wedge \zeta_1 + \psi_2 \wedge \zeta_2 + \cdots + \psi_n \wedge \zeta_n = \alpha(\pm 2b_{12} \pm 2a\alpha) w_1 \wedge w_2 = 0$$

because the factor $(\pm 2b_{12} \pm 2a\alpha) = 0$. Again, the normal bundle is flat.

Finally, we consider the case when the matrix A_S has one eigenvalue of multiplicity 3, i.e.,

$$A_S = \begin{pmatrix} a_o & 0 & -c & & & \\ 0 & a_o & c & & & \\ c & c & a_o & & & \\ & & & a_1 & & \\ & & & & \cdot & \\ & & & & & a_{n-3} \end{pmatrix}$$

with respect to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of the tangent space. Then just as in the cases above, we choose $w_1, \ldots, w_n$ so that $w_i(v_j) = \tau_i \delta_{ij}$. Then from A_S we read $A_S(v_i)$, $1 \le i \le n$, and use formula (5) to obtain

$$\begin{cases} \psi_1 = -a_o w_1 - c w_3, \\ \psi_2 = -a_o w_2 - c w_3, \\ \psi_3 = c w_1 - c w_2 - a_o w_3, \\ \psi_i = -a_{i-3} w_i, \quad i = 4, \ldots, n. \end{cases} \tag{10}$$

Assuming that the induced metric of the light cone representative f is flat, we get from the Gauss equation that $c = 0$, so this case does not occur. □

4. Branched channel Lorentzian hypersurfaces

In this section we show Theorem 1.1 which represents a version of the Cartan theorem [3,5,6] for conformally flat Lorentzian hypersurfaces in the projectivized light cone $\hat{\mathbb{R}}_1^{n+1}$ when $n \ge 4$. We begin with the natural extension of branched channel hypersurface to the Lorentzian setting.

Definition 4.1. A regular map $f : M_1^n \to \hat{\mathbb{R}}_1^{n+1}$ is called a branched channel hypersurface if it envelopes a spherical congruence S with rank $\mathrm{d}S \le 1$.

Before proving Theorem 1.1, we give some explicit examples: Since any two-dimensional pseudo-Riemannian manifold is conformally flat (see [2]), the next two examples represent conformally flat Lorentzian immersions in $\mathbb{R}_2^5$: Example 4.1 represents a branched channel surface in $\hat{\mathbb{R}}_1^3$ whose second fundamental form A_S has a null eigenvector, and Example 4.2 represents a conformally flat surface with second fundamental form having complex eigenvalues, which is not a branched channel hypersurface in $\hat{\mathbb{R}}_1^3$.

Example 4.1. Here is an example with $n = 2$ and $A_S^2 = 0$, $A_S \ne 0$. Let

$$f(x, y) = \left(x - y, x + y, y^2, \frac{1 - 4xy - y^4}{2}, \frac{1 + 4xy + y^4}{2} \right).$$

This is essentially the B-scroll put into the Möbius context. We can let

$$S(x, y) = (-y, -y, 1, y^2, -y^2)$$
$$S_1(x, y) = \left(-\sqrt{1 + y^2}, -\left(\frac{y^2}{\sqrt{1 + y^2}} \right), \frac{y}{\sqrt{1 + y^2}}, \frac{-x + y + y^3}{\sqrt{1 + y^2}}, \frac{x - y\left(1 + y^2\right)}{\sqrt{1 + y^2}} \right)$$
$$S_2(x, y) = \left(0, \frac{1}{\sqrt{1 + y^2}}, \frac{y}{\sqrt{1 + y^2}}, \frac{-x - y - y^3}{\sqrt{1 + y^2}}, \frac{x + y + y^3}{\sqrt{1 + y^2}} \right).$$

With this, the shape operator A_S is $\frac{1}{2(1+y^2)} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ in terms of the basis $\{S_1, S_2\}$. This has the form $\frac{1}{1+y^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with respect to a basis L_1, L_2 of null vectors.

Example 4.2. Here is an example with $n = 2$ and A_S, with complex eigenvalues. Let

$$f(x, y) = \left(\frac{\cos(x)\cosh(y)}{\sqrt{2}}, -\frac{\sin(x)\sinh(y)}{\sqrt{2}}, \frac{\cos(x)\sinh(y)}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\cosh(y)\sin(x)}{\sqrt{2}} \right).$$

This is a version of the complex circle $(\cos(z), \sin(z))$. We can let

$$\hat{f}(x,y) = \left(-\frac{\cos(x)\cosh(y)}{\sqrt{2}}, \frac{\sin(x)\sinh(y)}{\sqrt{2}}, -\frac{\cos(x)\sinh(y)}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{\cosh(y)\sin(x)}{\sqrt{2}}\right)$$

$$S(x,y) = (\sin(x)\sinh(y), \cos(x)\cosh(y), \cosh(y)\sin(x), 0, -\cos(x)\sinh(y))$$

$$S_1(x,y) = (-\cosh(y)\sin(x), -\cos(x)\sinh(y), -\sin(x)\sinh(y), 0, \cos(x)\cosh(y))$$

$$S_2(x,y) = (\cos(x)\sinh(y), -\cosh(y)\sin(x), \cos(x)\cosh(y), 0, \sin(x)\sinh(y)).$$

With this choice, the shape operator A_S is $\begin{pmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$ in terms of the basis $\{S_1, S_2\}$.

Following [5] we can begin with timelike surfaces of constant Gaussian curvature and find examples of three-dimensional conformally flat hypersurfaces in $\hat{\mathbb{R}}^4_1$, whose shape operators are not diagonalizable, or whose shape operator has complex eigenvalues.

Example 4.3. We begin with $p : U \subset \mathbb{R}^2_1 \to M^3_1(k)$, where the target space is either a sphere, plane or a hyperbolic space, and the image of p has constant Gaussian curvature and is not umbilic. For the first two cases we use the metric $-\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_4^2 + 2\mathrm{d}x_5\mathrm{d}x_6$. If $k > 0$, so that $\langle p, p\rangle = k$, the mapping is given by

$$f : U \times (0,\infty) \to \mathbb{R}^6_2 \quad f(t,s,r) = \left(p(t,s), \frac{1}{r\sqrt{2}}, -\frac{rk}{\sqrt{2}}\right) \hat{f}(t,r,s) = \frac{1}{2k}\left(p(t,s), -\frac{1}{r\sqrt{2}}, \frac{rk}{\sqrt{2}}\right).$$

The normal vector S given in terms of the normal n to the image of p in the sphere is $S(t,s,r) = (n,0,0)$.
When $k = 0$ we have

$$f : U \times \mathbb{R} \to \mathbb{R}^6_2 \quad f(x,y,t) = \left(p(x,y), t, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}(t^2 + \langle p,p\rangle)\right).$$

The normal vector S given in terms of the normal n to the image of p is

$$S(t,s,r) = (n, 0, 0, -\sqrt{2}\langle n, p\rangle).$$

For $k < 0$ we assume that the metric is $-\mathrm{d}x_1^2 - \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \cdots + \mathrm{d}x_6^2$ and use

$$f : U \times \mathbb{R} \to \mathbb{R}^6_2 \quad f(x,y,t) = (p(x,y), \sqrt{-k}\cos(t), \sqrt{-k}\sin(t)).$$

The normal vector S given in terms of the normal n to the image of p is $S(x,y,t) = (n,0,0)$.

As special cases we can take

$$f(x,y,t) = (\cos(x)\cosh(y), \sin(x)\cosh(y), \sin(x)\sinh(y), -\cos(x)\sinh(y), \cos(t), \sin(t)),$$

which has two complex eigenvalues and

$$\begin{aligned} f(u,v,t) = {} & \frac{1}{\sqrt{2}}((1+c)\sin(v), (1+c)\cos(v), (1-c)\sin(v), (1-c)\cos(v), 0, 0) \\ & + \frac{(u+cv)}{\sqrt{2}}(-\cos(v), \sin(v), \cos(v), \sin(v), 0, 0) \\ & + \left(0,0,0,0, \cos\left(\frac{t}{\sqrt{2c}}\right)\sqrt{2c}, \sin\left(\frac{t}{\sqrt{2c}}\right)\sqrt{2c}\right) \end{aligned}$$

whose shape operator is not diagonalizable.

The next result, for which we omit the proof, is analogous to the positive definite case, and shows that any branched channel hypersurface, $n \geq 3$, is conformally flat.

Proposition 4.1. *If M^n_1, $n \geq 3$, is a branched channel hypersurface then M^n_1 is conformally flat.*

Now we are interested in the converse of Proposition 4.1. For that we first establish the following lemma which is essentially Lemma 1.8.17 of [6]. We present a proof for dimensions $n \geq 4$ dealing with the two different types of shape operators.

Lemma 4.1. *A Lorentzian submanifold $f : M_1^n \to L_1^{n+2}$, $n \geq 4$, of the Lorentzian light cone defines a branched channel hypersurface if and only if its shape operator A_S with respect to some spacelike normal field $S : M_1^n \to S_2^{n+2}$ has $rkA_S \leq 1$.*

Proof. If rank $\mathrm{d}S \leq 1$ then rank $A_S \leq 1$. For the converse we use the vector field version of the Codazzi equation for $f : M_1^n \to \mathbb{R}_2^{n+3}$:

$$\nabla_X(A_S Y) - A_S(\nabla_X Y) - A_{\nabla_X^\perp S} Y = \nabla_Y(A_S X) - A_S(\nabla_Y X) - A_{\nabla_Y^\perp S} X.$$

Here we have $\nabla_X^\perp S = -\upsilon(X) f$ and $A_{\nabla_X^\perp S} Y = -\upsilon(X) Y$. We see from the structure equation that, for $Y \in \ker(A_S)$, $\mathrm{d}S(Y) = -\upsilon(Y) f$.

If X and Y are vector fields in $\ker(A_S)$ with $\langle X, X\rangle = \pm 1$ and $X \perp Y$ then, following [6], we have

$$\begin{aligned} \mathrm{d}S(Y) &= -\upsilon(Y) f = \mathrm{sgn}(\langle X, X\rangle)\langle -\upsilon(Y) X, X\rangle f \\ &= \mathrm{sgn}(\langle X, X\rangle)\langle A_S(\nabla_X Y) - \upsilon(X) Y - A_S(\nabla_Y X), X\rangle f = 0. \end{aligned}$$

If $\ker(A_S)$ is non-degenerate at a point and has dimension greater than or equal to 2, then for every vector field Y in the kernel, we can find the unit companion vector field X needed to show that $\mathrm{d}S(Y) = 0$.

On the other hand, if $\ker(A_S)$ is degenerate at a point x_o, we can find a vector field W so that $\ker(A_S) = W^\perp$ and $W(x_o)$ is null. The kernel is then spanned by $n-2$ spacelike orthonormal vector fields and an additional vector field V. We see that $\mathrm{d}S(V) = 0$ and if $n-2 \geq 2$ then the same is true for the spacelike vectors, but the proof fails if $n = 3$. □

So, making a change of the enveloped sphere congruence S via $\widetilde{S} = S + af$, with a being a function, one obtains that a Lorentzian hypersurface in $\hat{\mathbb{R}}_1^{n+1}$, $n \geq 4$, is a branched channel hypersurface if the Weingarten tensor field A_S with respect to any enveloped spherical congruence S has an eigenvalue of multiplicity $n-1$. We note that this definition makes sense for the non-diagonalizable A_S as well.

Theorem 4.1. *If $f : M_1^n \to \hat{\mathbb{R}}_1^{n+1}$, $n \geq 4$, is a conformally flat immersion then f is a branched channel hypersurface (i.e., A_S has* rank ≤ 1.*)*

Proof. Let F be an f-adapted frame of the flat lift of the conformally flat immersion in $\hat{\mathbb{R}}_1^{n+1}$, with connection form Φ. As usual, we analyze the three possible cases for the shape operator in the direction of S:

In the complex case we have the following equations from before Section 3:

$$\begin{aligned} &a_o^2 + b_o^2 + 2b_{11} = 0 \\ &a_o a + b_{11} + b = 0 \\ &-b_o a + b_{12} = 0 \\ &b_o a + b_{21} = 0 \\ &a^2 + 2b = 0, \quad \text{if } n \geq 4 \\ &b_{ij} = 0 \quad i \neq j, i, j \neq 1, 2. \end{aligned}$$

So we get Eqs. (7). Now we use Lemma 4.1, and change S to $\tilde{S} = S + af$. In this case, we see that the shape operator in the $\tilde{S}$-direction $A_{\tilde{S}} = A_S - aId$, i.e., $A_{\tilde{S}} = (\tilde{a}_{ij})_{n\times n}$ has the components $\tilde{a}_{11} = \tilde{a}_o = \tilde{a}_{22}$, $\tilde{a}_{12} = \tilde{b}_o = -\tilde{a}_{21}$ and $\tilde{a}_{ij} = 0$ for all $i, j \neq 1, 2$. But this means the equations are valid with $\tilde{}$'s. Thus, for $n \geq 4$ we get $\tilde{a} = 0 = \tilde{b}$, implying that $\tilde{b}_{11} = 0$ and finally that $\tilde{b}_0 = 0$, which means that the case cannot occur.

Next we examine the case where the shape operator is not diagonalizable. Here we know from Section 3 that the shape operator $A_S = (a_{ij})_{n\times n}$ has the components $a_{11} = \nu + \alpha$, $a_{22} = \nu - \alpha$, $a_{12} = -\alpha = -a_{21}$, $a_{ii} = a$ for $i \geq 3$, and all the others are zero. In this case we have the equations

$$\begin{aligned} &\nu^2 + b_{22} + b_{11} = 0 \\ &b_{1j} = 0 = b_{2j}, \qquad b_{j2} = 0 \quad \text{for } j > 2 \\ &a(\nu + \alpha) + b_{11} + b = 0 \\ &\alpha a + b_{12} = 0 \\ &b_{jk} = 0, \quad \text{for } j, k > 2, \end{aligned}$$

$$-\alpha a + b_{21} = 0$$
$$b_{j1} = 0, \quad \text{for } j > 2$$
$$a(\nu - \alpha) + b_{22} + b = 0$$
$$a^2 + 2b = 0 \quad \text{if } n \geq 4.$$

So we get (9). Using the same technique as in the complex case above, if $n \geq 4$ then $\tilde{b} = 0$, so that $\tilde{b}_{11} = 0 = \tilde{b}_{22} = \tilde{\nu}$ and the shape operator in the $\tilde{S}$-direction $A_{\tilde{S}} = (\tilde{a}_{ij})_{n\times n}$ has the components $\tilde{a}_{11} = \tilde{a}_{21} = \tilde{\alpha} = -\tilde{a}_{12} = -\tilde{a}_{22}$ and all the others are zero. So we get that $A_{\tilde{S}}$ has rank less than or equal to 1.

Finally, the real diagonalizable case follows in a similar fashion. $\square$

So, putting together Theorem 4.1 and Proposition 4.1, we have proved Theorem 1.1.

On the other hand, we observe that Theorem 4.1 does not hold in dimension $n = 3$, since Example 4.3 above represents three-dimensional conformally flat hypersurfaces in $\hat{\mathbb{R}}^4_1$ which are not branched channel hypersurfaces, being not diagonalizable or having two complex eigenvalues.

5. The three-dimensional case

In this section we are interested in studying the conformally flat Lorentzian hypersurfaces of dimension $n = 3$ in $\hat{\mathbb{R}}^4_1$. Some of the computations involved in this section are made in Appendix. Following [5,6], we define the Cartan tensor as (see Appendix A.2)

$$B = \sum_{i=1}^{3} \tau_i(\zeta_i - \sigma_i)v_i.$$

We know that B can be written as

$$B = -(trA_S)A_S + A_S^2 + \frac{1}{4}((trA_S)^2 - trA_S^2)Id,$$

for all algebraic types of the shape operator A_S. Studying the Cartan tensor and the condition that the Schouten tensor is a Codazzi tensor (see Appendix A.3), namely

$$\mathrm{d}\sigma_k - \sum_m \omega_{mk} \wedge \sigma_m = 0,$$

we get the following result:

Theorem 5.1. *An immersion $f : M^3_1 \to L^5_1$ is conformally flat if and only if*

$$d(-\tau_i\sigma_i + \tau_i\zeta_i) + \sum_j \omega_{ij} \wedge (-\tau_j\sigma_j + \tau_j\zeta_j) - \tau_i v \wedge \psi_i = 0. \tag{11}$$

We observe that, just as in the positive definite case, we can define $C = 2B + A_S^2$ and our conformal metric is $c(X, Y) = g(X, CY)$. In each algebraic case C has the same form with respect to the same basis.

Next we identify the conformal fundamental forms of the three-dimensional generic hypersurfaces in $\hat{\mathbb{R}}^4_1$ and prove, for those, the following local theorem.

Theorem 5.2. *$f : M^3_1 \to \hat{\mathbb{R}}^4_1$ is conformally flat iff the conformal fundamental forms are closed.*

For the proof we begin with the real case, i.e., we choose a frame which diagonalizes the Weingarten tensor field A_S of f with respect to S, that is, $\psi_i = -a_i w_i$. Because the proof is similar to the positive definite case, we simply note that a hypersurface $f : M^3_1 \to \hat{\mathbb{R}}^4_1$ with three distinct principal curvatures at all points is conformally flat if and only if the 1-forms

$$\gamma_k := \begin{cases} \sqrt{a_1 - a_2}\sqrt{a_1 - a_3}\omega_1 \\ \sqrt{a_1 - a_2}\sqrt{a_3 - a_2}\omega_2 \\ \sqrt{a_1 - a_3}\sqrt{a_2 - a_3}\omega_3, \end{cases}$$

where $\omega_i = \tau_i w_i$, are closed. In this case the conformal metric can be recovered as $-\gamma_1^2 + \gamma_2^2 + \gamma_3^2$.

For the other three cases we will use $\omega_i = \tau_i w_i$ in most of the equations. From now on we set $b_i = \zeta_i - \sigma_i$ so that

$$B = \sum_i \tau_i b_i v_i \tag{12}$$

Using Eq. (11) we get the three coordinate equations, namely,

$$-\mathrm{d}b_1 + \omega_{12} \wedge b_2 + \omega_{13} \wedge b_3 + v \wedge \psi_1 = 0 \tag{13}$$
$$\mathrm{d}b_2 - \omega_{12} \wedge b_1 + \omega_{23} \wedge b_3 - v \wedge \psi_2 = 0 \tag{14}$$
$$\mathrm{d}b_3 - \omega_{13} \wedge b_1 - \omega_{23} \wedge b_2 - v \wedge \psi_3 = 0. \tag{15}$$

Now consider the complex case. Using the three coordinate equations, the components b_i of the Cartan tensor in matrix form (26), Eqs. (6) and the first and second Codazzi equations (2), the coordinates equations reduce to

$$-2a_1 b_0 \mathrm{d}\omega_2 - 2b_0^2 \mathrm{d}\omega_1 + 2a_0 b_0 \mathrm{d}\omega_2 - d(a_1 b_0) \wedge \omega_2 - d(b_0^2) \wedge \omega_1 + d(a_0 b_0) \wedge \omega_2 = 0. \tag{16}$$
$$2a_1 b_0 \mathrm{d}\omega_1 - 2b_0^2 \mathrm{d}\omega_2 - 2a_0 b_0 \mathrm{d}\omega_1 + d(a_1 b_0) \wedge \omega_1 - d(b_0^2) \wedge \omega_2 - d(a_0 b_0) \wedge \omega_1 = 0. \tag{17}$$
$$(a_0^2 + b_0^2 - 2a_0 a_1 + a_1^2) \mathrm{d}\omega_3 + d(a_0^2/2 + b_0^2/2 - a_0 a_1 + a_1^2/2)\omega_3 = 0. \tag{18}$$

Then using Eqs. (16)–(18), one has that the 1-forms

$$\gamma_1 = \sqrt{2\mathrm{i} b_o (a_1 - a_o) - 2b_o^2}(\omega_1 + \mathrm{i}\omega_2)$$
$$\gamma_2 = \sqrt{2\mathrm{i} b_o (a_1 - a_o) + 2b_o^2}(\omega_1 - \mathrm{i}\omega_2)$$
$$\gamma_3 = \sqrt{a_o^2 + b_o^2 - 2a_0 a_1 + a_1^2}(\omega_3)$$

are closed. Conversely, if $\gamma_1, \gamma_2, \gamma_3$ above are closed it follows that Eqs. (16)–(18) hold. We observe that in this case the conformal metric can be recovered as $-\frac{\gamma_1^2}{2} + \frac{\gamma_2^2}{2} + \gamma_3^2$.

Next we look at the first non-diagonalizable case, i.e., that of multiplicity 2. Here we find

$$\psi_1 = (\nu + \alpha)\omega_1 - \alpha\omega_2$$
$$\psi_2 = -\alpha\omega_1 - (\nu - \alpha)\omega_2$$
$$\psi_3 = -a_1\omega_3,$$

and the components of the Cartan tensor B in matrix form are

$$b_1 = (\nu^2/2 + \alpha a_1)\omega_1 - \alpha a_1 \omega_2$$
$$b_2 = -\alpha a_1 \omega_1 + (\alpha a_1 - \nu^2/2)\omega_2$$
$$b_3 = (\nu^2/2 - \nu a_1)\omega_3.$$

So, substituting the 1-forms b_i and using the first and second Codazzi equations (2), we obtain that the first two equations become

$$d(\alpha(\nu - a_1)) \wedge \omega_1 + d(\alpha(a_1 - \nu)) \wedge \omega_2 + 2\alpha(\nu - a_1)\mathrm{d}\omega_1 + 2\alpha(a_1 - \nu)\mathrm{d}\omega_2 = 0,$$

implying that $\gamma_1 = \gamma_2 = \sqrt{2\alpha(\nu - a_1)}(\omega_1 - \omega_2)$ is a closed 1-form. In the same way the third equation gives

$$2(\nu - a_1)^2 \mathrm{d}\omega_3 + d((\nu - a_1)^2) \wedge \omega_3 = 0,$$

or $\gamma_3 = \sqrt{2(\nu - a_1)^2}\omega_3$ is a closed 1-form. The converse also holds. In addition, we note that in this case the conformal metric can be recovered as $-\gamma_1^2 + \frac{\gamma_3^2}{2}$.

Finally we look at the multiplicity 3 case. In the same way as before, we have

$$\psi_1 = a_o\omega_1 - c\omega_3$$
$$\psi_2 = -a_o\omega_2 - c\omega_3$$

$$\psi_3 = -c\omega_1 - c\omega_2 - a_0\omega_3$$
$$b_1 = \left(c^2 + \frac{a_o^2}{2}\right)\omega_1 + c^2\omega_2 - ca_0\omega_3$$
$$b_2 = c^2\omega_1 + \left(c^2 - \frac{a_o^2}{2}\right)\omega_2 - ca_0\omega_3$$
$$b_3 = -ca_o\omega_1 - ca_o\omega_2 - \frac{a_o^2}{2}\omega_3.$$

We look at the three coordinate equations again. The first gives:

$$c^2(\mathrm{d}\omega_1 + \mathrm{d}\omega_2) + \frac{1}{2}d(c^2) \wedge (\omega_1 + \omega_2) = 0 \tag{19}$$

or, equivalently, $\gamma_1 = c(\omega_1 + \omega_2)$ is a closed 1-form on M_1^3. The second equation gives the same result, while the third yields nothing. So, putting this all together, we have finished the proof of Theorem 5.2.

We now begin the proof of Theorem 1.2. We note that the integrability of the distributions is clear in the other three cases because if $\gamma_1 = \mathrm{d}x_1$, $\gamma_2 = \mathrm{d}x_2$ and we have a coordinate system $\{x_1, x_2, x_3\}$, then $\gamma_1 + \gamma_2 = 0$ is spanned by $\partial/\partial x_1 - \partial/\partial x_2$ and $\partial/\partial x_3$. We include more details in the complex case because the conformal fundamental forms are complex and involve a more complicated form of integrability. We want to prove that if M_1^3 is conformally flat then the umbilic distributions $\gamma_i \pm \gamma_j = 0$ are locally integrable in the complex case. We prove first that $\gamma_1 + \gamma_2 = 0$ is an integrable distribution and then that $\gamma_1 + \gamma_3 = 0$ is locally integrable, in the sense of Nirenberg [9,1].

We set $\sqrt{2\mathrm{i}b_o(a_1 - a_o) - 2b_o^2} = a + \mathrm{i}b$ and $c = \sqrt{(a_o - a_1)^2 + b_o^2}$. Then we have

$$\gamma_1 = (a\omega_1 - b\omega_2) + \mathrm{i}(b\omega_1 + a\omega_2)$$
$$\gamma_2 = (b\omega_1 + a\omega_2) + \mathrm{i}(a\omega_1 - b\omega_2)$$
$$\gamma_3 = c\omega_3$$

and we know that

$$d(a\omega_1 - b\omega_2) = 0, \qquad d(a\omega_2 + b\omega_1) = 0, \qquad d(c\omega_3) = 0. \tag{20}$$

Thus we have coordinates x, y, z such that

$$a\omega_1 - b\omega_2 = \mathrm{d}x, \qquad a\omega_2 + b\omega_1 = \mathrm{d}y, \qquad c\omega_3 = \mathrm{d}z. \tag{21}$$

$\gamma_1 + \gamma_2 = 0$ is equivalent to $\mathrm{d}x + \mathrm{d}y = 0$. Thus the distribution is spanned by $\{\partial/\partial x - \partial/\partial y, \partial/\partial z\}$ which is a real two-dimensional distribution. A basis for this distribution in the original eigenvectors is $\{(b - a)v_1 + (a + b)v_2, v_3\}$.

Next we consider $\gamma_1 + \gamma_3 = 0$ which is $\{\mathrm{d}x + \mathrm{i}\mathrm{d}y + \mathrm{d}z = 0\}$. A spanning set is given by $\{\partial/\partial x + \mathrm{i}\partial/\partial y, \partial/\partial x - \mathrm{i}\partial/\partial y - 2\partial/\partial z\}$ in the complexified tangent bundle of M, $T^{\mathbb{C}}$. In the original basis it can be written as $\{v_1 + \mathrm{i}v_2, c(a - \mathrm{i}b)(v_1 - \mathrm{i}v_2) - 2(a^2 + b^2)v_3\}$. Call this span $\mathcal{S}$. We can see easily that, in the terminology of Apostolova [1], $\mathcal{S}$ is formally integrable, meaning that $[\mathcal{S}, \mathcal{S}] \subset \mathcal{S}$, and, in addition, that $\mathcal{S} + \bar{\mathcal{S}}$ is also formally integrable. We can see easily that this second condition holds, because $\mathcal{S} + \bar{\mathcal{S}}$ is spanned by v_1, $\mathrm{i}v_2$ and v_3, and so is all of $T^{\mathbb{C}}$. Thus we have coordinates $\{z, y, t = (x - z)\}$ so that our distribution is dual to $\mathrm{d}z + \mathrm{i}\mathrm{d}y$ and $\mathrm{d}t$, exactly as one finds in [9]. The foliation associated to this system has leaves of the form $\mathbb{C} \times \mathbb{R}$ [12]. Finally, in a similar way one can see that the other distributions, i.e., $\gamma_2 + \gamma_3 = 0$ and $\gamma_i - \gamma_j = 0$, are locally integrable. So we have proved Theorem 1.2.

5.1. Guichard's nets

In the real, diagonalizable case and in the complex case just completed, the conformal fundamental forms give us three coordinates on our surface.

Indeed, in the real case we obtain a canonical coordinate system (x, y, z) such that $\gamma_1 = \mathrm{d}x$, $\gamma_2 = \mathrm{d}y$ and $\gamma_3 = \mathrm{d}z$, just as happens in the positive definite case. In particular, (x, y, z) are curvature line orthogonal coordinates and the coordinate surfaces $x = c_1$, $y = c_2$ and $z = c_3$, where c_1, c_2, c_3 are constants, constitute a triply orthogonal system.

In addition, one can see that defining $l_1 = \|\frac{\partial}{\partial x}\|$, $l_2 = \|\frac{\partial}{\partial y}\|$ and $l_3 = \|\frac{\partial}{\partial z}\|$, one obtains that $\sum_{i=1}^{3} l_i^2 = 0$, which is similar to the condition for a Guichard net in the positive definite case.

Now we turn to the complex version of Guichard's nets. Looking at the real and imaginary parts of the complex 1-forms γ_i we saw that there are coordinates x, y, z on M satisfying (21). We will see that the vector fields associated with the distributions $\{x + \mathrm{i}y, z\}$, $\{x - \mathrm{i}y, z\}$, and $\{x, y\}$ are the triply orthogonal distributions for the complex case.

These distributions are spanned by $\{\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$, $\{\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ and $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. Their normal vectors are eigenvectors for A. They can also be obtained by letting one of $x - \mathrm{i}y$, $x + \mathrm{i}y$, or z be constant.

To verify these claims, take the dual vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ for $\mathrm{d}x$ and $\mathrm{d}y$ and $\mathrm{d}z$ respectively. This means

$$\omega_1 = \frac{a}{a^2+b^2}\mathrm{d}x + \frac{b}{a^2+b^2}\mathrm{d}y, \qquad \omega_2 = -\frac{b}{a^2+b^2}\mathrm{d}x + \frac{a}{a^2+b^2}\mathrm{d}y, \qquad \omega_3 = \frac{1}{\sqrt{(a_o-a_1)^2+b_o^2}}\mathrm{d}z.$$

Equivalently we have

$$\frac{\partial}{\partial x} = \frac{a}{a^2+b^2}v_1 - \frac{b}{a^2+b^2}v_2, \qquad \frac{\partial}{\partial y} = \frac{b}{a^2+b^2}v_1 + \frac{a}{a^2+b^2}v_2, \qquad \frac{\partial}{\partial z} = cv_3.$$

Setting

$$l_1 = \frac{1}{\sqrt{2}(a+\mathrm{i}b)}, \qquad l_2 = \frac{1}{\sqrt{2}(\pm\mathrm{i})(a-\mathrm{i}b)}, \qquad l_3 = \frac{1}{c},$$

we have

$$I\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{b^2-a^2}{(a^2+b^2)^2} = l_2^2 - l_1^2 = -I\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right), \qquad I\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{c^2} = l_3^2,$$
$$I\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -\frac{2ab}{(a^2+b^2)^2} = -\mathrm{i}(l_1^2 + l_2^2), \qquad I\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = I\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = 0.$$

Now taking the complex vector fields $\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y})$, $\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$, we see that the dual 1-forms are $\frac{1}{\sqrt{2}}(\mathrm{d}x + \mathrm{i}\mathrm{d}y)$ and $\frac{1}{\sqrt{2}}(\mathrm{d}x - \mathrm{i}\mathrm{d}y)$ respectively, and these are orthogonal with lengths

$$I\left(\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\right) = -2l_1^2, \qquad I\left(\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right), \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)\right) = 2l_2^2.$$

Now taking our orthonormal basis v_1, v_2, v_3 where $\frac{v_1 - \mathrm{i}v_2}{\sqrt{2}}, \frac{v_1 + \mathrm{i}v_2}{\sqrt{2}}$ are eigenvectors of the shape operator A_S with eigenvalues $a_o + \mathrm{i}b_o$, $a_o - \mathrm{i}b_o$ respectively, we have

$$\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) = \sqrt{2}l_1\left(\frac{v_1 - \mathrm{i}v_2}{\sqrt{2}}\right), \qquad \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) = \mathrm{i}\sqrt{2}l_2\left(\frac{v_1 + \mathrm{i}v_2}{\sqrt{2}}\right),$$

and so these are eigenvectors of A_S. We have also

$$-2l_1^2 + 2l_2^2 - l_3^2 = 0,$$

which is similar to the condition for a Guichard net in the positive definite case.

To finish this section we note from Theorem 5.2 that, in the non-diagonalizable case of multiplicity 2, the 1-forms $\gamma_1 = \gamma_2 = \sqrt{2\alpha(\nu - a_1)}(\omega_1 - \omega_2)$ and $\gamma_3 = \sqrt{2(\nu - a_1)^2}\omega_3$, being closed, imply the existence of coordinates u, z so that $\mathrm{d}u = \sqrt{2\alpha(\nu - a_1)}(\omega_1 - \omega_2)$ and $\mathrm{d}z = \sqrt{2(\nu - a_1)^2}\omega_3$. One can see that the surfaces given by $u =$ constant and by $z =$ constant are perpendicular with normals given by $v_1 + v_2$ and v_3. In similar way, in the multiplicity 3 case one has that the 1-form $\gamma_1 = \gamma_2 = c(\omega_1 + \omega_2)$, being closed, defines a single coordinate v so that $\mathrm{d}v = c(\omega_1 + \omega_2)$.

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Appendix. Weyl conformal tensor in an indefinite metric

A.1. Weyl and Schouten tensors

We begin by setting

$$\rho_{ij} = \mathrm{d}\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj},$$

so that

$$R(X, v_k)v_i = \sum_j \tau_i \tau_j \rho_{ij}(X, v_k)v_j.$$

Defining the Ricci tensor as $\mathrm{Ric}(X, Y) = \mathrm{Tr}\{V \to R(X, V)Y\}$, we have

$$\mathrm{Ric}(X, Y) = \sum_{k=1}^{n} g(R(X, v_k)Y, v_k)\tau_k,$$

where $\{v_1, \ldots, v_n\}$ is our orthonormal basis. Then we let

$$\rho_i(X) = \mathrm{Ric}(X, v_i) = \sum_j \tau_i \tau_j \rho_{ij}(X, v_j). \tag{22}$$

Hertrich-Jeromin [5] defines the Schouten tensor as

$$s(X, v_i) = \sigma_i(X) = \frac{1}{n-2}\left(\rho_i(X) - \frac{\rho}{2(n-1)} g(X, v_i)\right) = \frac{1}{n-2}\left(\rho_i(X) - \frac{\rho}{2(n-1)} \tau_i \omega_i(X)\right). \tag{23}$$

The Weyl conformal tensor is defined as

$$\begin{aligned} W(X, Y, Z, W) = {} & R(X, Y, Z, W) - (s(X, Z)g(Y, W) - s(X, W)g(Y, Z) + g(X, Z)s(Y, W) \\ & - g(X, W)s(Y, Z)). \end{aligned}$$

Letting $Z = v_i$ and $W = v_j$ we get

$$W(X, Y, v_i, v_j) = \eta_{ij}(X, Y) = \rho_{ij}(X, Y)\tau_i - (\sigma_i \wedge \tau_j \omega_j + \tau_{\mathrm{i}} \omega_i \wedge \sigma_j)(X, Y).$$

Thus, if the Weyl tensor vanishes we have

$$\psi_i \wedge \psi_j + w_i \wedge (\zeta_j - \sigma_j) + (\zeta_i - \sigma_i) \wedge w_j = 0.$$

With these definitions we can prove the Weyl–Schouten Theorem in the indefinite setting [2]:

Theorem A.1. *An indefinite Riemannian manifold (M^n, g) of dimension $n \geq 3$ is conformally flat iff:*

1. *the Schouten tensor is a Codazzi tensor $\nabla_X s(Y, Z) = \nabla_Y s(X, Z)$ for $n = 3$;*
2. *the Weyl tensor vanishes if $n > 3$.*

A.2. The Cartan tensor

Now we are considering only $n = 3$. Following [5] (up to the sign and τ_j), we set

$$B = \sum_{i=1}^{3} \tau_i (\zeta_i - \sigma_i) v_i. \tag{24}$$

Lemma A.1. *For all algebraic types we have*

$$B = -(trA_S)A_S + A_S^2 + \frac{1}{4}((trA_S)^2 - trA_S^2)Id. \tag{25}$$

In each case we use the formulas for ψ_i and ζ_i to calculate ρ_i, ρ and σ_i. Using these values we can verify the form of B. We present the calculation for the complex case; the others are similar. In fact, in the complex case we will see that B has the matrix form above, i.e.,

$$B = \begin{pmatrix} -\dfrac{a_o^2+b_o^2}{2} & -a_1 b_o & \\ a_1 b_o & -\dfrac{a_o^2+b_o^2}{2} & \\ & & \dfrac{a_o^2+b_o^2}{2} - a_o a_1 \end{pmatrix}. \tag{26}$$

Using formulas (6) and (7) for ψ_i and ζ_i, we have $\rho_1(v_1) = -(a_o^2 + b_o^2 + 3b_{11} + a_o a_1 + b)$, $\rho_2(v_2) = a_o^2 + b_o^2 + 3b_{11} + a_o a_1 + b$, $\rho_3(v_3) = 2(b_{11} + a_o a_1 + b)$. So,

$$\rho/4 = (1/4)(-\rho_1(v_1) + \rho_2(v_2) + \rho_3(v_3)) = \frac{a_o^2+b_o^2}{2} + 2b_{11} + a_o a_1 + b.$$

Hence

$$\begin{aligned}
&\sigma_1(v_1) = \rho_1(v_1) + \rho/4 = -(a_o^2 + b_o^2)/2 - b_{11}\\
&\sigma_2(v_2) = \rho_2(v_2) - \rho/4 = (a_o^2 + b_o^2)/2 + b_{11}\\
&\sigma_3(v_3) = \rho_3(v_3) - \rho/4 = -(a_o^2 + b_o^2)/2 + a_o a_1 + b.\\
&\sigma_2(v_1) = \rho_2(v_1) = \sigma_1(v_2) = \rho_1(v_2) = b_{12} - a_1 b_o.\\
&\sigma_3(v_1) = \rho_3(v_1) = \sigma_1(v_3) = \rho_1(v_3) = 0.\\
&\sigma_2(v_3) = \rho_2(v_3) = \sigma_3(v_2) = \rho_3(v_2) = 0.
\end{aligned}$$

Using these values, we can see that $B(v_j) = \sum_{i=1}^{3} \tau_i(-\sigma_i(v_j) + \zeta_i(v_j))v_i$ has the correct form.

A.3. Equivalent condition for the Schouten tensor to be Codazzi

Next we would like to look at the condition for the Schouten tensor to be a Codazzi tensor, in other words, for

$$\nabla_X(s(Y,Z)) - s(\nabla_X Y, Z) - s(Y, \nabla_X Z) = \nabla_Y(s(X,Z)) - s(\nabla_Y X, Z) - s(X, \nabla_Y Z).$$

Using the notation above this is

$$\mathrm{d}\sigma_k - \sum_m \omega_{mk} \wedge \sigma_m = 0. \tag{27}$$

On the other hand, we also have $d\tau_k\zeta_k = \sum_{m=1}^{n} \tau_k\omega_{mk} \wedge \zeta_m + \tau_k v \wedge \psi_k$. Hence, assuming that the Schouten tensor is a Codazzi tensor, we see that

$$\nabla_X(BY) - B(\nabla_X Y) - \nabla_Y(BX) + B(\nabla_Y X) = \sum_i \tau_i(v \wedge \psi_i)(X,Y)v_i.$$

It follows that the Schouten tensor is a Codazzi tensor if and only if

$$d(-\tau_i\sigma_i + \tau_i\zeta_i) + \sum_j \omega_{ij} \wedge (-\tau_j\sigma_j + \tau_j\zeta_j) - \tau_i v \wedge \psi_i = 0. \tag{28}$$

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